Indian Statistical Institute, Bangalore Centre.
End-Semester Exam : Differential Equations

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Answer for 50 points. **Time Limit : 3 hours.**

Give necessary justifications and explanations for all your arguments.
If you are citing results from the class/books, mention it clearly.
Convergence of series needs to be justified.

1. Using Cauchy-Peano theorem or Picard-Lindelöf theorem, what can you say about the existence and uniqueness of local solutions for the following ODEs ? (**10**)

   (a) $y^{'} = 3(y-1)^{\frac{2}{3}}$; $y(0) = 1$.

   (b) $y^{'} = sgn(y)\sin(y)$; $y(0) = 0$. sgn denotes for the sign function.

   (c) $y^{'} = 1 + y^2$; $y(0) = 0$.

2. Let $p(x) \geq p_1(x) > 0$ and $q_1(x) \geq q(x)$ on the domain of the pair of differential equations

$$(p(x)u^{'})^{'} + q(x)u = 0,$$
$$(p_1(x)u_1^{'})^{'} + q_1(x)u_1 = 0.$$

   Assume that for all x with $v(x) \neq 0$, the following identity holds :

$$\left(\frac{u}{v}(p_1u'v - p_2uv')\right)' = (q_2 - q_1)\,u^2 + (p_1 - p_2)\,(u')^2 + p_2\left(u' - v'\frac{u}{v}\right)^2.$$

   Prove that between any two zeros of a non-trivial solution $u(x)$ of the first differential equation there lies at least one zero of every solution u_1 of the second differential equation (except in the trivial case of $u \equiv u_1, p \equiv p_1, q \equiv q_1$.) (**5**)

3. Conside the differential equation

$$y'' + \frac{p}{x^b}y' + \frac{q}{x^c}y = 0,$$

where p, q are non-zero real number and b, c are positive integers.

Explain what does one mean by a Frobenius series solution for a second-order homoeneous Linear ODE and show that if $b = 2, c = 3$ there is only one possible value of m for which there might exist a Frobenius series solution. (**5**)

4. Find the eigenvalues and eigenfunctions for the following boundary value problem

$$y'' + \lambda y = 0 \ ; \ y(L) - y(-L) = 0 \ ; \ y'(L) - y'(-L) = 0.$$

Is this a regular Sturm-Liouville boundary value problem ? What properties of the eigenfunctions or eigenvalues differ from those of a regular Sturm-Liouville boundary value problem ? (**8**)

5. Consider the following system of first order linear equations :

$$\mathbf{x}'(t) = \begin{pmatrix} \alpha & 2 \\ -2 & 0 \end{pmatrix} \mathbf{x}(t),$$

where $\alpha \in \mathbb{R} \setminus \{-4, 4\}$. Describe the general solution for the above equation in terms of α ? In this case, what can you say about the qualitative behaviour of the solutions as $t \to \infty$ for different α ?(**8**)

6. **You can answer at most three out of the following four problems. Each question carries 8 points.** (**24**)

   (a) Solve the following PDE via method of characteristics :

   $$-xu_t + tu_x = 0, \ u(x, 0) = cosx^2, x \in \mathbb{R}, t > 0.$$

   (b) Let $\Omega \subset \mathbb{R}^n$ be a bounded, open, connected subset for $n \geq 1$. Show that the following PDE has a unique solution.

   $$\Delta U(x) = 0 \ x \in \Omega; \ U(x) = \phi(x) \ x \in \partial\Omega,$$

   where $\partial\Omega$ denotes the boundary of Ω and is also assumed to be connnected and compact. Further, you may assume that $U \in C^2(\Omega) \cap C(\bar{\Omega})$, $\phi(.) \in C(\partial\Omega)$.

(c) Consider a rod of length L with uniform cross-section. Let the axis of the bar lie along x-axis from $x = 0$ to $x = L$. Assume further that the temperature distribution u is only a function of the axial co-ordinate x and time t (i.e., the cross-sectional dimensions are negligible). For boundary conditions, assume the ends of the bar to be insulated i.e., there is no flow of heat through them. Also, at time $t = 0$, the temperature distribution is $sin(\frac{\pi x}{L})$ where $0 \leq x \leq L$.

i. Write down the PDE corresponding to the above heat conduction problem along with the boundary conditions. (You may use the fact that the rate of heat flow across a cross-section is proportional to the rate of change of temperature in the x direction).

ii. Find the solution to the above PDE with constants derived explicitly.

iii. Find the steady-state temperature i.e., $u(x, t)$ as $t \to \infty$?

Note: In (i), you are required to only write down the PDE and not derive it. But provide justification for the choice of boundary conditions.

(d) Consider the general wave equation

$$a^2 u_{xx} = u_{tt}, \; u(x, 0) = f(x), \; u_t(x, 0) = g(x), \; x \in (0, L), t > 0,$$

$$u(0, t) = u(L, t) = 0, \; t > 0.$$

Find the solution to the wave equation by either one of the following two methods

METHOD 1 - Seperation of variables **OR**

METHOD 2 - First show that every solution to $a^2 u_{xx} = u_{tt}$ is of the form $\psi(x - at) + \phi(x + at)$ for two functions ψ, ϕ.